

ON POLYGONAL MEASURES WITH VANISHING HARMONIC MOMENTS

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ABSTRACT. A signed polygonal measure is the sum of finitely many real constant density measures supported on polygons. Given a finite set $S \subset \mathbb{C}$, we study the existence of signed polygonal measures spanned by polygons with vertices in S , which have all harmonic moments vanishing. For S generic, we show that the dimension of the linear space of such measures is $\binom{|S|-3}{2}$.

We also investigate the situation where the resulting density is either 0 or ± 1 , which corresponds to pairs of polygons of unit density having the same logarithmic potential at ∞ . We show that such a signed measure does not exist if $|S| \leq 5$, but for each $n \geq 6$ there exists an S , with $|S| = n$, giving rise to such a signed measure.

1. INTRODUCTION

Inverse problems in logarithmic potential theory have attracted substantial attention since the publication of the fundamental paper [15], where P.S. Novikov, in particular, proved that two convex (or, more generally, star-shaped) domains in \mathbb{C} with unit density cannot have the same logarithmic potential near ∞ . Notice that the knowledge of the germ of a logarithmic potential of a finite compactly supported Borel measure μ at ∞ is equivalent to the knowledge of the sequence of its harmonic moments $m_j(\mu)$, $j = 0, 1, \dots$, where the j -th harmonic moment of μ is defined by:

$$m_j(\mu) = \int_{\mathbb{C}} z^j d\mu.$$

More precisely, if

$$u_\mu(z) := \int_{\mathbb{C}} \ln |z - \xi| d\mu(\xi)$$

is the logarithmic potential of μ and

$$\mathfrak{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi} = \frac{\partial u_\mu(z)}{\partial z}$$

is its Cauchy transform then the Taylor expansion of $\mathfrak{C}_\mu(z)$ at ∞ has the form:

$$\mathfrak{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \dots$$

Thus Novikov's result can be reformulated as the statement that two convex domains in \mathbb{C} with unit density cannot have coinciding sequences of harmonic moments. It is well-known that already for non-convex domains with unit density the uniqueness in this problem no longer holds. For instance, examples of pairs of non-convex polygons with the same logarithmic potential near ∞ can be found on [6, p. 333], see Fig. 2 below. By a *convex polygon* we mean the convex hull of finite many points in the plane at least 3 of which are non-collinear. A general *polygon* is the union of finitely many convex polygons. By a *vertex* of a polygon we mean a

Date: September 19, 2012.

2010 Mathematics Subject Classification. Primary 44A60; Secondary 31B20.

Key words and phrases. potential theory, harmonic moments, polygonal measures.

point of its boundary such that its sufficiently small ϵ -neighborhood in the polygon is different from a half-disk of radius ϵ .

The class of general polygons as well as domains bounded by lemniscates has attracted a substantial attention in this area. Several authors have also considered the class of polynomial densities instead of the unit density.

Given a domain $D \subset \mathbb{C}$, define its *standard measure*

$$\mu_D = \chi_D dx dy,$$

where χ_D is the characteristic function of D . We say that two polygons in \mathbb{C} are *equipotential* if their standard measures create coinciding logarithmic potential outside their union.

Notice that if different polygons with constant (but not necessarily unit) density have the same logarithmic potential near ∞ then they must have the same set of vertices, see [6, Corollary 2 and Lemma 2]. (The coincidence of the logarithmic potential near ∞ implies even more restrictions on the polygons than just the coincidence of their set of vertices, cf. [6].)

Taking this fact into account we pose the following *classical inverse logarithmic potential problem for polygons in \mathbb{C}* .

Problem 1. Given a finite set $S \subset \mathbb{C}$, determine whether there exist two equipotential polygons whose sets of vertices coincide with S .

One can show that for generic S no pairs of equipotential polygons exist.

Definition 1. A complex (respectively, real) *polygonal (signed) measure* $\mu := \mu(\mathcal{D})$ is the sum

$$\mu := \sum_{D \in \mathcal{D}} c_D \mu_D, \quad |\mathcal{D}| < \infty, \quad c_D \in \mathbb{C} \text{ (respectively, } c_D \in \mathbb{R}),$$

where \mathcal{D} is a finite set of closed and bounded polygons. By a *node* of μ we understand a vertex of $D \in \mathcal{D}$.

The densities c_D might be different for different $D \in \mathcal{D}$, and D and $D' \in \mathcal{D}$ may overlap. Another choice of \mathcal{D} can still lead to the same μ , in turn leading to a different set of nodes.

Besides the nodes of μ it is natural to talk about the *vertices* of μ , which are $v \in \mathbb{C}$ such that for any sufficiently small $\epsilon > 0$ and any $v' \in \mathbb{C}$ s.t. $\|v - v'\|^2 = 4\epsilon$ one has

$$\int_{\|v-z\|^2 \leq \epsilon} d\mu(z) \neq \int_{\|v'-z\|^2 \leq \epsilon} d\mu(z).$$

The idea behind this is to distinguish v from points which lie in the center of a disk, where μ has either constant density, or two different densities in two halves of the disk. Obviously, the set of vertices of μ is a subset of the set of intersections of sides of the polygons in \mathcal{D} . There exists a finite collection $\tilde{\mathcal{D}}$ of connected, closed and bounded polygons with pairwise empty intersections of interiors, such that $\mu = \mu(\tilde{\mathcal{D}})$, and nodes and vertices of μ coincide. However, such a representation of μ need not be the most economic one, see e.g. Example 3. It turns out that nodes provide a more natural set of parameters in our setting.

Assume that S admits a pair of equipotential polygons. Considering the difference of their standard measures, one obtains a polygonal signed measure supported on the convex closure $\text{conv}(S)$ of S with density attaining only values $0, \pm 1$ and with all harmonic moments vanishing. Conversely, if one can find a polygonal signed measure with all vanishing harmonic moments and such that its density attains only values $0, \pm 1$, then one obtains a pair of equipotential polygons by taking the differences of $\text{conv}(S)$ with the sets where the density attains value 1 and -1 , respectively.

If we weaken the condition that the density of a polygonal signed measure attains only values $0, \pm 1$ then we arrive at the setup of the present paper. Given a *spanning* set S (i.e. S contains at least 3 non-collinear points), we introduce the linear spaces $\mathfrak{M}^{\mathbb{R}}(S) \subset \mathfrak{M}^{\mathbb{C}}(S)$ of real-valued and complex-valued polygonal measures obtained as the real and the complex linear spans of the standard measures of all general polygons with nodes in S . (Obviously, $\mathfrak{M}^{\mathbb{C}}(S) = \mathbb{C} \otimes \mathfrak{M}^{\mathbb{R}}(S)$.) Since each polygon can be triangulated on its set of nodes, the latter spaces can be defined as the linear spans of the standard measures of all triangles with nodes in S .

In this note we make a further step in the study of (non-)uniqueness in logarithmic potential theory by considering the following question.

Problem 2. Given a finite set $S \subset \mathbb{C}$, determine the linear subspace $\mathfrak{M}_{null}^{\mathbb{R}}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ of signed real-valued polygonal measures (resp. of signed complex-valued polygonal measures $\mathfrak{M}_{null}^{\mathbb{C}}(S) \subset \mathfrak{M}^{\mathbb{C}}(S)$) all whose harmonic moments are vanishing.

The main technical tool we use is the *normalized generating function* $\Psi_{\mu}(u)$ for *harmonic moments* of a measure μ , defined by

$$\Psi_{\mu}(u) = \sum_{j=0}^{\infty} \binom{j+2}{2} m_j(\mu) u^j. \quad (1.1)$$

Notice that $\Psi_{\mu}(u)$ is closely related to the Cauchy transform $\mathfrak{C}_{\mu}(z)$ at ∞ . Namely,

$$\Psi_{\mu}(u) = \frac{1}{2} \frac{d^2}{du^2} \left(\sum_{j=0}^{\infty} m_j(\mu) u^{j+2} \right).$$

At the same time for a compactly supported measure μ and sufficiently large $|z|$, $z\mathfrak{C}_{\mu}(z) = \sum_{j=0}^{\infty} m_j(\mu)/z^j$. Thus for $|u|$ sufficiently small we get

$$\Psi_{\mu}(u) = \frac{1}{2} \frac{d^2}{du^2} \left(u \mathfrak{C}_{\mu} \left(\frac{1}{u} \right) \right).$$

Similar multivariate generating functions were recently considered in [14]. Important in our consideration are the following observations.

Proposition 1. For measures μ with compact support one has

$$\Psi_{\mu}(u) = \sum_{j=0}^{\infty} \binom{j+2}{2} m_j(\mu) u^j = \int \frac{d\mu}{(1-uz)^3}. \quad (1.2)$$

The normalized generating function $\Psi_{\Delta}(u)$ of (the standard measure of) the triangle $\Delta \subset \mathbb{C}$ whose vertices are located at a, b, c is given by

$$\Psi_{\Delta}(u) = \frac{\text{Area } \Delta}{(1-au)(1-bu)(1-cu)}.$$

Note that the integral transform in (1.2) appears to be a variant of *Fantappiè transformation*, cf. [4].

Definition 2. We say that a finite set $S = \{z_0, z_1, \dots, z_n\}$ of points in \mathbb{C} is *non-degenerate* if no three of its points are collinear.

Proposition 2. For any non-degenerate set $S = \{z_0, z_1, \dots, z_n\}$, $n \geq 2$ of points in \mathbb{C} and any fixed non-negative integer $j \leq n$, the set of (standard measures of) all triangles with a node at z_j is a basis of the spaces $\mathfrak{M}^{\mathbb{R}}(S)$ and $\mathfrak{M}^{\mathbb{C}}(S)$. In particular,

$$\dim_{\mathbb{R}} \mathfrak{M}^{\mathbb{R}}(S) = \dim_{\mathbb{C}} \mathfrak{M}^{\mathbb{C}}(S) = \binom{n}{2}.$$

We are interested in linear subspaces $\mathfrak{M}_{null}^{\mathbb{R}}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ (resp. $\mathfrak{M}_{null}^{\mathbb{C}}(S) \subset \mathfrak{M}^{\mathbb{C}}(S)$) of real-valued (resp. complex-valued) measures having all vanishing harmonic moments.

The main results of this paper are as follows.

Proposition 3. For any non-degenerate set $S = \{z_0, z_1, \dots, z_n\}$, $n \geq 2$ of points in \mathbb{C} ,

$$\dim_{\mathbb{C}} \mathfrak{M}_{null}^{\mathbb{C}} = \binom{n-1}{2}.$$

Example 1. For $n = 3$ the space $\mathfrak{M}_{null}^{\mathbb{C}}(S)$ is spanned by the complex-valued measure $\tilde{\mu}$ whose densities with respect to the basis of triangles $\Delta_{012}, \Delta_{013}, \Delta_{023}$ are given by:

$$\begin{cases} d_{012} = (z_1 - z_2)/|[012]| \\ d_{013} = (z_3 - z_1)/|[013]| \\ d_{023} = (z_2 - z_3)/|[023]| \end{cases},$$

where $[i, j, k] = \det \begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix}$ stands for twice the signed area of the triangle with nodes z_i, z_j, z_k and $z_j = x_j + y_j I$, I being the imaginary unit.

Remark 1. For S non-degenerate, the space $\mathfrak{M}_{null}^{\mathbb{C}}(S)$ projects isomorphically on the linear subspace of $\mathfrak{M}^{\mathbb{C}}(S)$ spanned by all triangles $\Delta_{0,i,j}$ where $2 \leq i < j \leq n$. In other words, assigning arbitrarily complex-valued densities $d_{0,i,j}$, $2 \leq i < j \leq n$ we can uniquely determine the densities $d_{0,1,j}$, $j = 2, \dots, n$ to get a measure belonging to $\mathfrak{M}_{null}^{\mathbb{C}}(S)$.

Theorem 1. For any non-degenerate set $S = \{z_0, z_1, \dots, z_n\}$, $n \geq 2$ of points in \mathbb{C} ,

$$\dim_{\mathbb{R}} \mathfrak{M}_{null}^{\mathbb{R}}(S) = \binom{n-2}{2}.$$

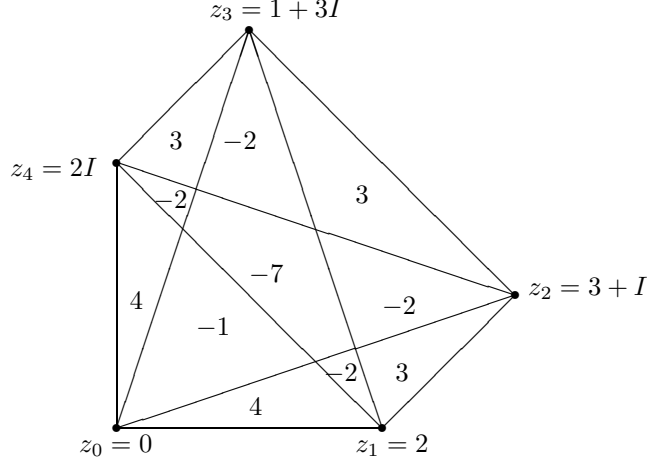
Remark 2. For S non-degenerate, the space $\mathfrak{M}_{null}^{\mathbb{R}}(S)$ projects isomorphically on the linear subspace of $\mathfrak{M}^{\mathbb{R}}(S)$ spanned by all triangles $\Delta_{0,i,j}$ where $3 \leq i < j \leq n$. In other words, arbitrarily real-valued densities $d_{0,i,j}$, $3 \leq i < j \leq n$, uniquely determine the densities $d_{0,1,j}$, $j = 2, \dots, n$ and $d_{0,2,j}$, $j = 3, \dots, n$ of a measure belonging to $\mathfrak{M}_{null}^{\mathbb{R}}(S)$.

Theorem 2. For any non-degenerate 5-tuple $S = \{z_0, z_1, z_2, z_3, z_4\}$, the space $\mathfrak{M}_{null}^{\mathbb{R}}(S)$ is spanned by the real measure $\tilde{\mu}$ with densities with respect to the basis of triangles $\Delta_{012}, \Delta_{013}, \Delta_{014}, \Delta_{023}, \Delta_{024}, \Delta_{034}$ given by:

$$\begin{cases} d_{012} = ||z_1 - z_2||^2 [134][234]/|[012]| \\ d_{013} = -||z_1 - z_3||^2 [124][234]/|[013]| \\ d_{014} = ||z_1 - z_4||^2 [123][234]/|[014]| \\ d_{023} = ||z_2 - z_3||^2 [124][134]/|[023]| \\ d_{024} = -||z_2 - z_4||^2 [134][123]/|[024]| \\ d_{034} = ||z_3 - z_4||^2 [123][124]/|[034]| \end{cases} \quad (1.3)$$

Example 2. For the 5-tuple $\{0, 2, 3+I, 1+3I, 2I\}$ the measure $3\tilde{\mu}$ is shown in Fig. 1 below. (In this case $3\tilde{\mu}$ has integer densities which are easier to show \TeX nically.)

Remark 3. Notice that knowing the densities of a polygonal measure $\mu \in \mathfrak{M}_{null}^{\mathbb{R}}(S)$ with respect to the basic triangles containing a fixed node (say z_0) we still need to find the densities in all its chambers where by a *chamber* we mean a connected

FIGURE 1. Measure $3\tilde{\mu}$ spanning $\mathfrak{M}_{null}^{\mathbb{R}}(0, 2, 3 + I, 1 + 3I, 2I)$.

component of $\text{conv}(S) \setminus \text{Arr}(S)$, $\text{Arr}(S)$ being the union of all lines connecting pairs of points in S . (Integers in Fig. 1 show the densities in the chambers they are placed in.) Each chamber is contained in a number of basic triangles and the density of a given chamber equals the sum of the densities of all basic triangles containing it. Containment of chambers in triangles (and more generally in simplices in \mathbb{R}^d) can be coded by an appropriate incidence matrix whose rows correspond to simplices and columns correspond to chambers. If a simplex contains a chamber then the corresponding entry equals 1, otherwise the entry equals 0. This incidence matrix of chambers and simplices in \mathbb{R}^d was for the first time studied in [3] and later in [1, 2]. It has rather delicate properties and already the number of chambers is a complicated function of the initial non-degenerate set S . In particular, this number can change if we deform S within the class of non-degenerate sets. This circumstance partially explains why results of the present paper do not solve the classical Problem 1.

Remark 4. Notice that if $S = \{z_0, \dots, z_n\}$ consists of complex numbers having only rational real and imaginary parts then one can choose a basis of $\mathfrak{M}_{null}^{\mathbb{R}}(S)$ consisting of polygonal measures with integer densities.

To continue, we need the following example of a pair of equipotential polygons.

Example 3. In [6] one finds pairs of non-convex polygons with the same logarithmic potential near ∞ , from which measures $\tilde{\mu} \in \mathfrak{M}_{null}^{\mathbb{R}}(S)$ can be obtained. E.g. the corresponding $\tilde{\mu}$ from [6, Example 1] can be described as follows. Consider the 6-tuples $T = \{\pm\sqrt{3} \pm I, \pm 2I\}$ and $T' = \{\pm \frac{1 \pm \sqrt{3}I}{2}, \pm 1\}$. Let $F \subset \mathbb{C}$ be the difference of the convex hull of T and the union of the set of 6 triangles obtained as the orbit of the triangle with nodes $(\sqrt{3} + I, \sqrt{3} - I, 1)$ under the rotation by $\frac{\pi}{3}$, see Fig. 2. Let $F' \subset \mathbb{C}$ be the difference of the convex hulls of T and of T' . Then $\Psi_{\mu_F}(u) = \Psi_{\mu_{F'}}(u)$, as F and F' have the same logarithmic potential. Therefore $\tilde{\mu} := \mu_F - \mu_{F'} \in \mathfrak{M}_{null}^{\mathbb{R}}(T \cup T')$.

Observe that $\tilde{\mu} \in \mathfrak{M}_{null}^{\mathbb{R}}(T)$, although the polygons themselves have 12 vertices! This illustrates the non-uniqueness of representation of $\tilde{\mu}$. Indeed,

$$2\tilde{\mu} = \sum_{0 \leq j \leq 5} \mu_{\exp(\frac{i\pi I}{3})(\sqrt{3}+I, \sqrt{3}-I, -2I)} - \sum_{0 \leq j \leq 1} \mu_{(\sqrt{3}+(-1)^j I, \sqrt{3}-(-1)^j I, -\sqrt{3}+(-1)^j I)}.$$

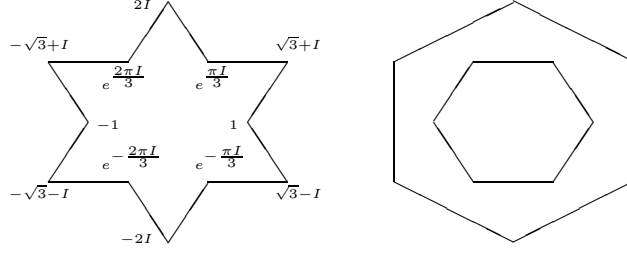


FIGURE 2. Example of a pair of equipotential polygons.

Using the above example together with Theorem 2 we can prove the following result related to the classical Problem 1.

Theorem 3. For each $n \geq 6$ there exist S , with $|S| = n$, admitting a pair of equipotential polygons. No such S exists if $|S| \leq 5$.

The essential part of the proof of Theorem 3 is to deal with the case $|S| = 5$.

Our final result concerns a natural cone spanned by the standard measures of triangles with nodes in S . Namely, for an arbitrary non-degenerate set $S = \{z_0, z_1, \dots, z_n\}$ denote by $\mathfrak{K}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ the $\binom{n}{2}$ -dimensional cone obtained by taking non-negative linear combinations of the standard measures of all triangles with nodes in S . (Recall that $\mathfrak{M}^{\mathbb{R}}(S)$ is the linear span of these measures.)

Theorem 4. Extremal rays of $\mathfrak{K}(S)$ are spanned by (the standard measures) of triangles which do not contain any point of S different from its own nodes. In particular, if S is a convex configuration, (i.e. each z_j belongs to the convex hull of S) then every triangle with nodes in S spans an extremal ray of $\mathfrak{K}(S)$.

We finish the introduction with a conjectural description of all faces of $\mathfrak{K}(S)$. We say that a pair of triangles with vertices in S *forms a flip* if they have a common side and their convex hull is a 4-gon. With any pair of triangles forming a flip we associate their *flipped pair* obtained by removing the opposite diagonal from their convex hull, see Case a) Fig.2 below. (On this figure the pairs of triangles $(\Delta_{013}, \Delta_{123})$ and $(\Delta_{012}, \Delta_{023})$ form a flip and each pair is the flipped one to the other pair.)

Conjecture 1. A collection Col of triangles having no internal vertices spans a face of $\mathfrak{K}(S)$ if and only if for each pair of triangles from Col forming a flip its flipped pair of triangles is also contained in Col .

The necessity of the stated condition is quite obvious and its sufficiency might follow from the results of [3].

Acknowledgements. B.S. is grateful to the Division of Mathematical Sciences of Nanyang Technological University for hospitality in April 2012 when this project was carried out. D.V.P. is supported by Singapore MOE Tier 2 Grant MOE2011-T2-1-090 (ARC 19/11). The authors thank Sinai Robins for helpful discussions.

2. PROOFS

Proof of Proposition 1. First, we prove (1.2). Indeed,

$$\int \frac{d\mu(z)}{(1-uz)^3} = \sum_{k \geq 0} u^k \int \binom{k+2}{2} z^k d\mu(z) = \sum_{k \geq 0} t^k \binom{k+2}{2} m_k(\mu) = \Psi_\mu(u),$$

as required. By [7, (1)], for any $f(z)$ analytic in the closure of Δ , we have

$$\frac{1}{2\text{Area}\Delta} \int_{\Delta} f''(z) dx dy = \sum_{k=1, j \neq i \in \{1,2,3\} \setminus \{k\}}^k \frac{f(z_k)}{(z_k - z_i)(z_k - z_j)}.$$

Applying the latter identity and (1.2) to $f(z) = \frac{1}{2u^2} \frac{1}{1-uz}$, we get the claimed formula. \square

To prove Proposition 2 we need to recall some basic notions. First we present a description of all linear dependences among the standard measures of all triangles with vertices in a non-degenerate set S . Namely, any 4-tuple of points (say, $\{z_0, z_1, z_2, z_3\}$) in S has 4 triangles with vertices at these points. To study linear dependences between these 4 triangles, one has to distinguish between two cases. Consider the convex hull of $\{z_0, z_1, z_2, z_3\}$, which is either a quadrangle or a triangle, see Fig. 2. Obviously, in Case a) we have (up to permutation of the vertices) the equality $\mu_{\Delta_{013}} + \mu_{\Delta_{123}} = \mu_{\Delta_{023}} + \mu_{\Delta_{012}}$. Analogously, in Case b) we have (up to permutation of the vertices) the relation $\mu_{\Delta_{012}} = \mu_{\Delta_{013}} + \mu_{\Delta_{123}} + \mu_{\Delta_{023}}$.

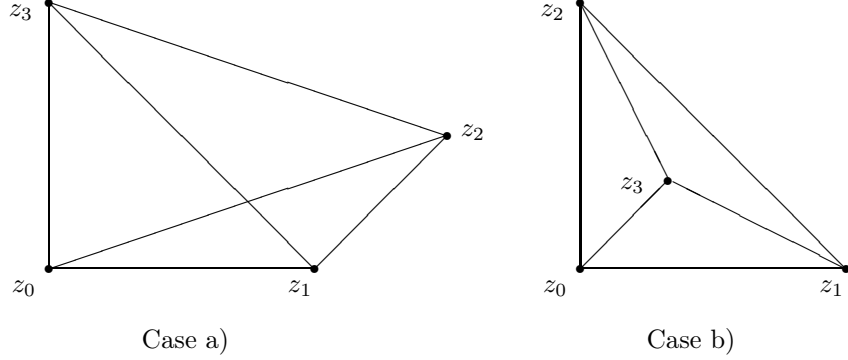


FIGURE 3. Linear dependence of 4 triangles spanned by 4 points.

The next statement is a special case of [3, Theorem 1]. (Unfortunately, it seems that a proof of this important statement is missing in the available literature.) J.A. De Loera informed us that it can be derived from results in [8] or [9].

Proposition 4. Linear dependences among the standard measures of all triangles with vertices in S are generated by the above linear dependences coming from all possible 4-tuples of vertices in S .

Using Proposition 4 we immediately obtain that the set of (the standard measures of) all triangles containing a given vertex $z_j \in S$ spans $\mathfrak{M}^{\mathbb{R}}(S)$. To complete the proof of Proposition 2 we need to show that if S is non-degenerate then the latter measures are linearly independent. We need more notions.

Definition 3. By a 2-chain $\mathcal{C}^{(2)}$ we mean a formal linear combination

$$\mathcal{C}^{(2)} = \alpha_1 \Delta_1 + \alpha_2 \Delta_2 + \dots + \alpha_s \Delta_s \quad (2.1)$$

of triangles $\Delta_1, \dots, \Delta_s$ in \mathbb{C} with real or complex coefficients where each triangle is equipped with the standard orientation induced from \mathbb{C} .

By using the standard pairing

$$\langle f dx dy, \mathcal{C}^{(2)} \rangle = \int_{\mathcal{C}^{(2)}} f dx dy = \sum_{j=1}^s \alpha_j \int_{\Delta_j} f dx dy,$$

one sees that a 2-chain (2.1) defines a linear functional on the space $\Omega^{(2)}$ of smooth 2-forms on \mathbb{C} .

Definition 4. Analogously, by a 1-chain $\mathcal{C}^{(1)}$ we mean a formal linear combination

$$\mathcal{C}^{(1)} = \beta_1 I_1 + \beta_2 I_2 + \dots + \beta_t I_t \quad (2.2)$$

of oriented finite interval I_1, \dots, I_s in \mathbb{C} with real or complex coefficients.

Again, by using the standard pairing

$$\langle \omega, \mathcal{C}^{(1)} \rangle = \int_{\mathcal{C}^{(1)}} \omega = \sum_{j=1}^t \beta_j \int_{I_j} \omega,$$

one sees that a 1-chain (2.2) defines a linear functional on the space $\Omega^{(1)}$ of smooth 1-forms on \mathbb{C} .

Definition 5. For a given triangle Δ with vertices a, b, c where triple (a, b, c) is counterclockwise oriented we define its *boundary* $\partial\Delta$ as the sum of three oriented intervals $[ab] + [bc] + [ca]$. As usual, we extend by linearity the boundary operator ∂ to the linear space of all 2-chains.

Definition 6. A 2-chain (resp. a 1-chain) is called *vanishing* if it defines the zero linear functional on $\Omega^{(2)}$ (resp. $\Omega^{(1)}$).

Lemma 1. A 2-chain $\mathcal{C}^{(2)}$ is vanishing if and only if its boundary $\partial\mathcal{C}^{(2)}$ is a vanishing 1-chain.

Proof. Stokes theorem says that $\int_{\partial\Delta} w = \int_{\Delta} dw$, where $w \in \Omega^{(1)}$, Δ is an arbitrary triangle, $\partial\Delta$ is its boundary and dw is the differential of w . (Recall that if $w = F(x, y)dx + G(x, y)dy$ then $dw = (F'_x + G'_y)dxdy$.) For any 2-form $f(x, y)dxdy$ we can represent it as dw_x where $w_x = F(x, y)dx$ and $F(x, y)$ is the primitive function of $f(x, y)$ along any horizontal line. Analogously, $f(x, y)dxdy$ equals dw_y where $w_y = G(x, y)dy$ and $G(x, y)$ is the primitive function of $f(x, y)$ along any vertical line. Thus

$$\int_{\mathcal{C}^{(2)}} f dxdy = \int_{\partial\mathcal{C}^{(2)}} w_x = \int_{\partial\mathcal{C}^{(2)}} w_y.$$

If the l.h.s. vanishes for all $f dxdy$ then $\partial\mathcal{C}^{(2)}$ should vanish and vice versa. \square

Proof of Proposition 2. We need to show that the standard measures of all triangles containing z_0 are linearly independent. Indeed, by Lemma 1 a 2-chain $\mathcal{C}^{(2)}$ of triangles vanishes if and only if its boundary chain $\partial\mathcal{C}^{(2)}$ vanishes. But if S is non-degenerate then each triangle $\Delta_{0,i,j}$ has its unique edge (z_i, z_j) in the boundary and no chain of the form $\beta_{i,j}(z_i, z_j)$ with non-trivial $\beta_{i,j}$ can be vanishing. Therefore the standard measures of triangles $\Delta_{0,i,j}$ form a basis in $\mathfrak{M}^{\mathbb{C}}(S)$ and $\mathfrak{M}^{\mathbb{R}}(S)$. \square

Proof of Proposition 3. The case $n = 2$ is trivial, so we assume $n \geq 3$. Given a non-degenerate $S = \{z_0, z_1, \dots, z_n\}$, consider the complex-valued measure μ obtained by assigning (complex) densities d_{0ij} , $1 \leq i < j \leq n$ to triangles Δ_{0ij} . Set $m_{i,j} = d_{0ij} \text{Area } \Delta_{0ij}$. Then the normalized generating function $\Psi_{\mu}(u)$ for harmonic moments of μ is given by

$$\begin{aligned} \Psi_{\mu}(u) &= \sum_{1 \leq i < j \leq n} d_{0ij} \Psi_{\Delta_{0ij}}(u) = \sum_{1 \leq i < j \leq n} \frac{m_{ij}}{(1 - z_0 u)(1 - z_i u)(1 - z_j u)} \\ &= \frac{1}{1 - z_0 u} \frac{P(u)}{\prod_{i=1}^n (1 - z_i u)}, \end{aligned} \quad (2.3)$$

where $P(u)$ is a polynomial of degree at most $n - 2$. Its coefficients at $1, u, u^2, \dots, u^{n-2}$ are the consecutive entries of the vector $\mathcal{M}_n^{\mathbb{C}} \cdot \mathbf{m}_n$ of length $(n - 1)$, where

$$\mathbf{m}_n = (m_{12}, m_{13}, \dots, m_{n-1,n})^{\top}$$

with $m_{i,j}$'s ordered lexicographically and $\mathcal{M}_n^{\mathbb{C}}$ is the $(n - 1) \times \binom{n}{2}$ -matrix with columns corresponding to $m_{i,j}$; such a column contains consecutive elementary symmetric functions of the $(n - 2)$ -tuple $(z_1, z_2, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_n)$, where \hat{z}_i and \hat{z}_j stands for the omission of these points.

Example 4. For $n = 4$ the coefficients at $(1, u, u^2)$ of the numerator $P(u)$ of (2.3) are the consecutive entries of the vector $\mathcal{M}_4^{\mathbb{C}} \cdot \mathbf{m}_4$ where

$$\mathbf{m}_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^{\top} \quad \text{and}$$

$$\mathcal{M}_4^{\mathbb{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ z_3 + z_4 & z_2 + z_4 & z_2 + z_3 & z_1 + z_4 & z_1 + z_3 & z_1 + z_2 \\ z_3 z_4 & z_2 z_4 & z_2 z_3 & z_1 z_4 & z_1 z_3 & z_1 z_2 \end{pmatrix}.$$

In other words,

$$P(u) = \sum_{\substack{1 \leq i < j \leq 4 \\ k < \ell, \{i,j\} \cap \{k,\ell\} = \emptyset}} (m_{ij} + (z_i + z_j)m_{k\ell}u + z_i z_j m_{k\ell}u^2)$$

Consider the maximal minor $\text{Min}_n^{\mathbb{C}}$ of $\mathcal{M}_n^{\mathbb{C}}$ formed by the columns corresponding to $m_{12}, \dots, m_{1,n}$, i.e. the first $n - 1$ columns of $\mathcal{M}_n^{\mathbb{C}}$.

Lemma 2. $\det_n = \det(\text{Min}_n^{\mathbb{C}}) = \prod_{2 \leq i < j \leq n} (z_i - z_j)$.

Proof. Indeed, the degree of $\det(\text{Min}_n^{\mathbb{C}})$ as a polynomial in z_2, \dots, z_n equals $\binom{n-1}{2}$. We need to show that it vanishes if and only if $z_i = z_j$. The 'if' part is obvious since the column corresponding to $m_{1,i}$ will coincide with the column corresponding to $m_{1,j}$. To see the remaining part, argue by contradiction and assume that $(\alpha_{12}, \dots, \alpha_{1n})$ is a nontrivial linear dependence among the columns of $\text{Min}_n^{\mathbb{C}}$. The $1k$ -th column consists of the coefficients of the polynomial $g_{1k}(u) = \frac{\prod_{j=1}^n (1 - z_j u)}{1 - z_k u}$, and our linear dependence is a linear dependence among such polynomials. Evaluate these at $\frac{1}{z_j}$ and note that $g_{1k}(\frac{1}{z_j})$ vanish whenever $k \neq j$. Thus $\alpha_{1j} = 0$, a contradiction. Thus $\det(\text{Min}_n^{\mathbb{C}})$ is divisible by $\prod_{2 \leq i < j \leq n} (z_i - z_j)$. Substituting $z_2 = 0, z_3 = 1, \dots, z_n = n - 2$ we can check that the normalizing factor equals 1. \square

Remark 5. By using Cramer's rule, is it not difficult to give an explicit formula for the inverse $(\text{Min}_n^{\mathbb{C}})^{-1}$.

From Lemma 2 we know that for any $S = \{z_0, z_1, \dots, z_n\}$ with pairwise distinct (not necessarily non-degenerate) points the rank of $\mathcal{M}_n^{\mathbb{C}}$ equals $n - 1$. Thus the kernel of $\mathcal{M}_n^{\mathbb{C}}$ which by definition coincides with $\mathfrak{M}_{n\text{null}}^{\mathbb{C}}(S)$ has dimension $\binom{n}{2} - (n - 1) = \binom{n-1}{2}$. \square

Proof of Theorem 1. The space $\mathfrak{M}_{n\text{null}}^{\mathbb{R}}(S) \subset \mathfrak{M}_{n\text{null}}^{\mathbb{C}}(S)$ is the maximal by inclusion real subspace of the complex kernel. In other words, it can be interpreted as the kernel of the matrix $\mathcal{M}_n^{\mathbb{R}}$ obtained by taking the real and imaginary parts of all rows of $\mathcal{M}_n^{\mathbb{C}}$.

The case $n = 2$ is trivial. The case $n = 3$ can be dealt with by explicitly computing the kernel of $\mathfrak{M}_3^{\mathbb{C}}$ and seeing that it does not contain real vectors if S is non-degenerate. Thus we assume $n \geq 4$. Since the first row of $\mathcal{M}_n^{\mathbb{C}}$ equals $(1, 1, \dots, 1)$ the matrix $\mathcal{M}_n^{\mathbb{R}}$ has size $(2n - 3)\binom{n}{2}$, see (2.4). Ordering m_{ij} 's lexicographically, consider the maximal minor $\text{Min}_n^{\mathbb{R}}$ of $\mathcal{M}_n^{\mathbb{R}}$ formed by the columns

corresponding to $(2n - 3)$ variables $m_{12}, m_{13}, \dots, m_{1n}, m_{23}, m_{24}, \dots, m_{2n}$, i.e. the first $(2n - 3)$ columns of $\mathcal{M}_n^{\mathbb{R}}$.

Lemma 3. $\det \text{Min}_n^{\mathbb{R}} = C[123][124] \cdots [12n] \prod_{3 \leq i < j \leq n} |z_i - z_j|^2$, $0 \neq C \in \mathbb{R}$.

Proof. We begin by showing that $\Theta := \det \text{Min}_n^{\mathbb{R}}$ is divisible by $[12k]$ for any $3 \leq k \leq n$. As $[12k]$ is an irreducible quadratic polynomial in x_1, x_2, x_k and y_1, y_2, y_k , it suffices to show that vanishing of $[12k]$ implies vanishing of Θ . Vanishing of $[12k]$ is equivalent to existence of $a \in \mathbb{R}$ satisfying $z_k = az_1 + (1 - a)z_2$. The latter implies that $\text{Min}_n^{\mathbb{R}}$ has linearly dependent columns 12, $1k$, and $2k$. Indeed, they consist, respectively, of the coefficients of

$$\begin{aligned} g_{12}(u) &= (1 - az_1u - (1 - a)z_2u) \times (1 - z_3u) \cdots (1 - z_{k-1}u)(1 - z_{k+1}u) \cdots (1 - z_nu) \\ g_{1k}(u) &= (1 - z_2u) \times (1 - z_3u) \cdots (1 - z_{k-1}u)(1 - z_{k+1}u) \cdots (1 - z_nu) \\ g_{2k}(u) &= (1 - z_1u) \times (1 - z_3u) \cdots (1 - z_{k-1}u)(1 - z_{k+1}u) \cdots (1 - z_nu) \end{aligned}$$

which are linearly dependent: $g_{12}(u) = (a - 1)g_{1k}(u) - ag_{2k}(u)$.

To show that Θ is divisible by $|z_i - z_j|^2 = (z_i - z_j)(\bar{z}_i - \bar{z}_j)$ for any $3 \leq i < j \leq n$, observe that $z_i = z_j$ implies $g_{ki}(u) = g_{kj}(u)$ for $k = 1, 2$.

It remains to see that Θ is not identically 0. Arguing by contradiction, let $(\alpha_{12}, \alpha_{13}, \dots, \alpha_{1n}, \alpha_{23}, \dots, \alpha_{2n})$ be the coefficients of a nontrivial real linear dependence among the columns of $\text{Min}_n^{\mathbb{R}}$. The latter columns correspond to the coefficients of $g_{ij}(u)$. Evaluating these at $u = \frac{1}{z_k}$, for $3 \leq k \leq n$, makes all of them but g_{1k} and g_{2k} vanish. Thus

$$\alpha_{1k}g_{1k}(z_k^{-1}) + \alpha_{2k}g_{2k}(z_k^{-1}) = 0, \quad \text{implying } \alpha_{1k} = -\alpha_{2k} \frac{z_k - z_2}{z_k - z_1} \quad \text{and} \quad \frac{z_k - z_2}{z_k - z_1} \in \mathbb{R}.$$

A direct computation shows that the rightmost relation is equivalent to $[12k] = 0$, a contradiction. \square

Lemma 3 implies that for any non-degenerate S the matrix $\mathcal{M}_n^{\mathbb{R}}$ has rank equal to $2n - 3$. Therefore, $\dim \mathfrak{M}_{null}^{\mathbb{R}}(S) = \binom{n}{2} - (2n - 3) = \binom{n-2}{2}$. \square

Proof of Theorem 2. For $S = \{0, z_1, z_2, z_3, z_4\}$ the space $\mathfrak{M}_{null}^{\mathbb{R}}(S)$ is given by the system

$$\begin{aligned} \mathcal{M}_4^{\mathbb{R}} \cdot \mathbf{m}_4 &= 0, \quad \text{where } \mathbf{m}_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^{\top} \text{ and} \\ \mathcal{M}_4^{\mathbb{R}} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_3 + x_4 & x_2 + x_4 & x_2 + x_3 & x_1 + x_4 & x_1 + x_3 & x_1 + x_2 \\ y_3 + y_4 & y_2 + y_4 & y_2 + y_3 & y_1 + y_4 & y_1 + y_3 & y_1 + y_2 \\ x_3x_4 - y_3y_4 & x_2x_4 - y_2y_4 & x_2x_3 - y_2y_3 & x_1x_4 - y_1y_4 & x_1x_3 - y_1y_3 & x_1x_2 - y_1y_2 \\ x_3y_4 + x_4y_3 & x_2y_4 + x_4y_2 & x_2y_3 + x_3y_2 & x_1y_4 + x_4y_1 & x_1y_3 + x_3y_1 & x_1y_2 + x_2y_1 \end{pmatrix}. \end{aligned} \quad (2.4)$$

Recall that a $k \times (k + 1)$ -matrix T of rank k has right kernel spanned by the vector $(T^{(1)}, \dots, T^{(k+1)})$, where $T^{(j)}$ is the minor of T with j th column removed multiplied by $(-1)^j$. Thus (2.4) has a unique (up to a scaling) solution of the form:

$$\begin{cases} m_{12} = & |z_1 - z_2|^2 [134][234] \\ m_{13} = - & |z_1 - z_3|^2 [124][234] \\ m_{14} = & |z_1 - z_4|^2 [123][234] \\ m_{23} = & |z_2 - z_3|^2 [124][134] \\ m_{24} = - & |z_2 - z_4|^2 [134][123] \\ m_{34} = & |z_3 - z_4|^2 [123][124] \end{cases}$$

It is easy to prove this. We give a sketch here. Note that m_{12} equals to the (negative of the) determinant of the matrix $\mathcal{M}_4^{\mathbb{R}}$ with 1st column removed (we

denote it by $A^{(12)}$). Then, $\det A^{(12)}$ is divisible by $\|z_1 - z_2\|^2$, as the rank of $A^{(12)}$ drops when $z_1 = z_2$, and as $\|z_1 - z_2\|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ is the product of two irreducible polynomials with complex coefficients.

Similarly, $\det A^{(12)}$ is divisible by $[234]$ (and a very similar argument applies to $[134]$). To see this, note that, as $[234]$ is irreducible, it suffices to show that its vanishing implies vanishing of $\det A^{(12)}$. To this end, assume that $z_4 = az_2 + (1 - a)z_3$, and make this substitution in $A^{(12)}$. Then its last 3 columns become linearly dependent (with coefficients $(1, a - 1, -a)$):

$$\begin{pmatrix} 1 & 1 & 1 \\ ax_2 - ax_3 + x_1 + x_3 & x_1 + x_3 & x_1 + x_2 \\ ay_2 - ay_3 + y_1 + y_3 & y_1 + y_3 & y_1 + y_2 \\ ax_1x_2 - ax_1x_3 - ay_1y_2 + ay_1y_3 + x_1x_3 - y_1y_3 & x_1x_3 - y_1y_3 & x_1x_2 - y_1y_2 \\ ax_1y_2 - ax_1y_3 + ax_2y_1 - ax_3y_1 + x_1y_3 + x_3y_1 & x_1y_3 + x_3y_1 & x_1y_2 + x_2y_1 \end{pmatrix}.$$

□

Proof of Theorem 3. To prove the first part, we recall that Example 3 settles the case $|S| = 6$. To settle the case $|S| = 6 + q$, we modify the latter Example. Add q points P_1, \dots, P_q outside $\text{conv}(T)$, so that P_1, \dots, P_q and $\sqrt{3} \pm I$ are in the convex position, and Q is the resulting convex $q + 2$. Then $F \cup Q$ and $F' \cup Q$ are equipotential $6 + q$ -gons, by additivity of the measure.

To prove the second part, we have consider the cases $|S| = 3, 4, 5$, one by one. Cases $|S| = 3, 4$ follow from Theorem 1.

It remains to deal with the only non-trivial case $|S| = 5$. We have to consider the incidence matrices between the chambers and the basic simplices for all possible non-degenerate 5-tuples of points S . One can easily see that for non-degenerate 5-tuples there are (up to permutation of the vertices) only 3 different cases to consider depending on the shape of $\text{conv}(S)$ which can be a 5-gon, a 4-gon, or a triangle. The corresponding incidence matrices $\text{Inc}_1, \text{Inc}_2, \text{Inc}_3$ are given below using the labeling presented in Figures 4 and 5 for these cases. (Greek letters in Fig. 4 denote the vertices of the inner 5-gon. They will be needed below.) We show that in none of these case one can find a pair of equipotential polygons.

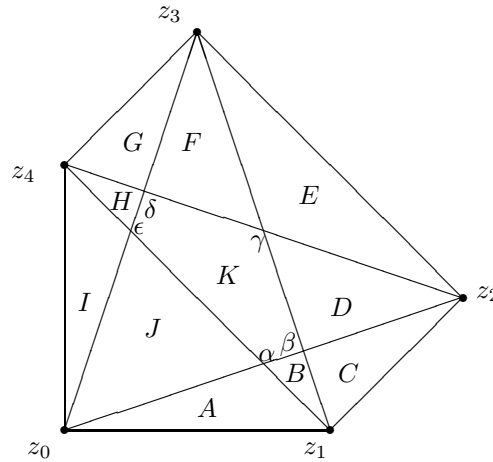


FIGURE 4. Chambers and their labeling for $\text{conv}(S)$ a 5-gon.

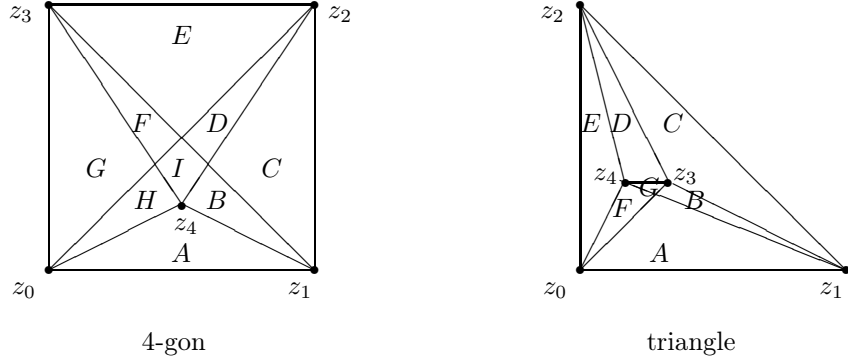


FIGURE 5. Chambers and their labeling for $\text{conv}(S)$ a 4-gon or a triangle.

$$\begin{aligned}
 \text{Inc}_1 = & \begin{matrix} & A & B & C & D & E & F & G & H & I & J & K \\ \begin{matrix} \Delta_{012} \\ \Delta_{013} \\ \Delta_{014} \\ \Delta_{023} \\ \Delta_{024} \\ \Delta_{034} \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \\
 \text{Inc}_2 = & \begin{matrix} & A & B & C & D & E & F & G & H & I \\ \begin{matrix} \Delta_{012} \\ \Delta_{013} \\ \Delta_{014} \\ \Delta_{023} \\ \Delta_{024} \\ \Delta_{034} \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}, \quad \text{Inc}_3 = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} \Delta_{012} \\ \Delta_{013} \\ \Delta_{014} \\ \Delta_{023} \\ \Delta_{024} \\ \Delta_{034} \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}.
 \end{aligned}$$

For brevity, we introduce notation $\frac{1}{2}||[K]||$ for the area of a polygon K . First, we need an elementary

Lemma 4. For an arbitrary triangle $\Delta_{\alpha\beta\gamma}$ and arbitrary secants $\alpha\epsilon$, $\beta\delta$, see Fig. 6 the area of triangle $\Delta_{\alpha\beta\zeta}$ is bigger than that of $\Delta_{\epsilon\delta\zeta}$:

$$||[\Delta_{\alpha\beta\zeta}]|| > ||[\Delta_{\epsilon\delta\zeta}]||.$$

Proof. Indeed, draw the line $\alpha\kappa$ parallel to $\beta\gamma$ and extend $\beta\delta$ till it hits $\alpha\kappa$. (The intersection point of the latter lines is denoted by η .) Triangles $\Delta_{\alpha\beta\zeta}$ and $\Delta_{\eta\epsilon\zeta}$ have equal area. Indeed, they are obtained from $\Delta_{\alpha\beta\eta}$ and $\Delta_{\alpha\epsilon\eta}$, respectively, by removing $\Delta_{\alpha\zeta\eta}$. Notice that $\Delta_{\alpha\beta\eta}$ and $\Delta_{\alpha\epsilon\eta}$ have the same base $\alpha\eta$ and equal heights. \square

Case 2. Using labeling on the left part of Fig. 5 and (1.3) we conclude that densities $d_{012}, d_{014}, d_{023}, d_{034}$ are positive while d_{013}, d_{024} are negative. From chambers E and C we conclude $d_{023} = d_{012} = 1$. Then from chamber D we have that either $d_{024} = -1$ or -2 . The second case leads to $d_{013} = 0$, contradiction. Thus $d_{024} = -1$ which from chamber I gives $d_{013} = -1$. Finally, $d_{034} = d_{014} = 1$. Thus chambers A, C, E, G have density 1, chamber I has density -1 and the remaining chambers have vanishing density. We need to show that $||[I]|| < ||[A]|| + ||[C]|| + ||[E]|| + ||[G]||$. We will show that actually $||[I]|| < ||[C]|| + ||[G]||$. Cut I into two triangles by drawing its diagonal connecting z_4 with non-neighboring vertex p of I (lying strictly above above z_4 of Fig. 5). Extending z_3p and z_0z_4 we get a triangle containing G and the left half of I and we can apply Lemma 4. Analogously, extending z_2p and z_1z_4

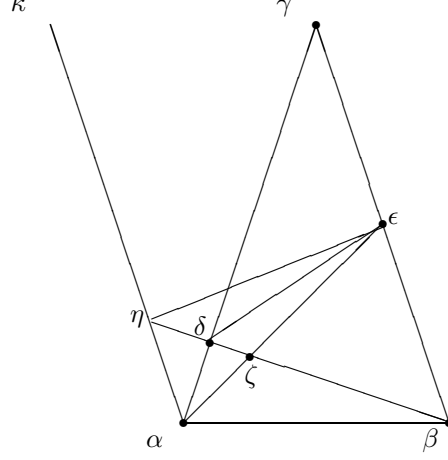


FIGURE 6. Illustration to Lemma 4

we get a triangle containing C and the right half of I and we can apply Lemma 4. Thus the required measure does not exist.

Case 3. Using labeling on the right part of Fig. 5 and formulas (1.3) we again conclude that densities $d_{012}, d_{014}, d_{024}$ are negative, while $d_{013}, d_{023}, d_{034}$ are positive. Similar considerations as above give $d_{012} = d_{014} = d_{024} = -1$ and $d_{013} = d_{023} = d_{034} = 1$. Thus, the densities of A, C, E are -1 and the density of G is 1 . In fact, $||G|| < ||A||$ already. Indeed, extending the interval $z_0 z_4$ and $z_1 z_3$ till they intersect at a point, say p we get the triangle $z_0 z_1 p$ to which we apply Lemma 4. Thus the required measure does not exist.

Case 1. Using labeling on Fig. 4 and (1.3) we see that densities $d_{012}, d_{014}, d_{023}, d_{034}$ are positive while d_{013}, d_{024} are negative. Assuming that the densities of all chambers attain only values $0, \pm 1$ and looking at chambers C, E, G we get that $d_{023} = d_{012} = d_{034} = 1$. Looking at chamber D we conclude that $d_{024} = -1$. (It might be equal -2 as well but then looking at chamber K we have to conclude that $d_{013} = 0$ which is impossible.) From chamber K we get $d_{013} = -1$ and from chamber I we get $d_{014} = 1$. Thus the density in chambers A, C, E, G, I equals 1 , in chamber K it equals -1 and it vanishes in the remaining chambers. Notice that the total mass of the measure should vanish. To see that this cannot happen, we show that $||K|| < ||A|| + ||C|| + ||E|| + ||G|| + ||I||$. Using Lemma 4 we conclude that $||A|| > ||\Delta_{\alpha\beta\epsilon}||$, $||C|| > ||\Delta_{\alpha\beta\gamma}||$, $||E|| > ||\Delta_{\beta\gamma\delta}||$, $||G|| > ||\Delta_{\delta\epsilon\gamma}||$, and $||I|| > ||\Delta_{\epsilon\alpha\delta}||$, see Fig. 4. Triangles $\Delta_{\alpha\beta\epsilon}, \Delta_{\alpha\beta\gamma}, \Delta_{\beta\gamma\delta}, \Delta_{\delta\epsilon\gamma}, \Delta_{\epsilon\alpha\delta}$ pairwise overlap. These overlapping consists of 5 smaller triangles inside K . The complement in K to the union of triangles $\Delta_{\alpha\beta\epsilon}, \Delta_{\alpha\beta\gamma}, \Delta_{\beta\gamma\delta}, \Delta_{\delta\epsilon\gamma}, \Delta_{\epsilon\alpha\delta}$ is a small 5-gon inside K . Now we can use these 5 small triangles to cover the small 5-gon inside K . We get exactly the same situation as the original one and we can apply the same argument as we did and cover a substantial part of the small 5-gon etc. Continuing this process we will in infinitely many steps exhaust the original 5-gon K . Thus the required measure does not exist. \square

To prove Theorem 4 we need the following observation.

Lemma 5. The convex hull of the standard measures of 4 triangles as in Case a) Fig. 3, i.e. two pairs forming a flip is a plane quadrangle. The convex hull of the standard measures of 4 triangles as in Case b) Fig. 3 is a plane triangle.

Proof. Obvious from the relations given above Fig. 3. \square

Proof of Theorem 4. Indeed if a triangle Δ contains an interior point other than its vertices then μ_Δ is the sum of three triangles in which it is subdivided by an inner vertex, see Lemma 5. (Recall that S is non-degenerate by assumption.) Thus μ_Δ is not an extremal ray. On the other hand, assume that no point in S other than its vertices is contained in Δ and μ_Δ is a linear combination of the standard measures of some other triangles with vertices in S with positive coefficients. Since no such triangle can be contained strictly inside Δ by assumption and all coefficients are positive we get that any such linear combination necessarily has positive density somewhere outside Δ , contradiction. \square

3. OPEN PROBLEMS

1. Theorem 1 gives the dimension of $\mathfrak{M}_{null}^{\mathbb{R}}(S)$ for non-degenerate S . Its dimension for arbitrary S is unclear. On one hand, if S is degenerate then $\dim \mathfrak{M}^{\mathbb{R}}(S)$ decreases. On the other hand, the number of equations impose on the densities might also decrease. It seems highly plausible that $\dim \mathfrak{M}_{null}^{\mathbb{R}}(S)$ for an arbitrary S depends only on non-oriented matroid associated to this set, see e.g. [10]. An algorithm calculating this dimension is given in [2].

2. Besides the cone $\mathfrak{K}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ one can introduce a more important, bigger, cone $\mathfrak{K}_{pos}(S) \subset \mathfrak{M}^{\mathbb{R}}(S)$ where $\mathfrak{K}_{pos}(S) \supset \mathfrak{K}(S)$ consists of all non-negative measures from $\mathfrak{M}^{\mathbb{R}}(S)$.

Conjecture 2. The combinatorial structure of $\mathfrak{K}_{pos}(S)$ depends only on the oriented matroid associated to S .

Already for generic configurations S with 6 points the combinatorial structure of $\mathfrak{K}_{pos}(S)$ and, in particular, the set of its extremal rays seems to be quite complicated. We plan to study this fascinating subject in the future.

3. Notice that we have a natural linear map $\Psi_\mu : \mathfrak{M}^{\mathbb{R}}(S) \rightarrow Rat_n$ obtained by associating to each measure $\mu \in \mathfrak{M}^{\mathbb{R}}(S)$ its normalized generating function (1.1). Here Rat_n is the linear space of rational functions of the form $R(u) = \frac{P(u)}{\prod_{j=0}^n (1 - z_j u)}$, $\deg P(u) \leq n - 2$ having real constant term. Obviously, $\dim Rat_n = 2n - 3$ and using Theorem 1 we obtain that $\mathfrak{M}^{\mathbb{R}}(S)$ is mapped onto Rat_n . The following question is very natural in connection with the inverse problem for the class of non-negative measures.

Problem 3. Describe the extremal rays/faces of the image cones $\Psi_\mu(\mathfrak{K}(S))$ and $\Psi_\mu(\mathfrak{K}_{pos}(S))$ in Rat_n .

4. We have an example of a pair of equipotential polygons with $|S| = 6$, see Fig. 2.

Problem 4. Describe all 6-tuples S admitting a pair of equipotential polygons.

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